

DEFICIENCY AND ABELIANIZED DEFICIENCY OF SOME VIRTUALLY FREE GROUPS

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ABSTRACT. Let Q_m be the HNN extension of $\mathbf{Z}/m \times \mathbf{Z}/m$ where the stable letter conjugates the first factor to the second. We explore small presentations of the groups $\Gamma_{m,n} = Q_m * Q_n$. We show that for certain choices of (m, n) , for example $(2, 3)$, the group $\Gamma_{m,n}$ has a relation gap unless it admits a presentation with at most 3 defining relations, and we establish restrictions on the possible form of such a presentation. We then associate to each (m, n) a 3-complex with 16 cells. This 3-complex is a counterexample to the $D(2)$ conjecture if $\Gamma_{m,n}$ has a relation gap.

1. INTRODUCTION

Given a finite presentation F/R for a group Γ , the action of the free group F by conjugation on R induces an action of Γ on the abelian group $R_{\text{ab}} = R/[R, R]$. It is obvious that the rank of R_{ab} as a $\mathbf{Z}\Gamma$ -module is at most the number of elements needed to generate R as a normal subgroup of F ; it therefore serves as a lower bound on the minimal number of relators needed to present Γ on the given generators. This lower bound seems extremely crude: one can hardly believe that it will be sharp in general. And yet, despite sustained attack over many years, not a single example has emerged to lend substance to this intuition.

A finite presentation F/R whose *relation module* R_{ab} has rank strictly smaller than the number of elements required to generate R as a normal subgroup of F is said to have a *relation gap*; the *relation gap problem* is to determine whether or not there exists a presentation with a relation gap. It belongs to a circle of famous and notoriously hard problems concerning the homotopy properties of finite 2-complexes. For example, it is closely related to the $D(2)$ conjecture.

In [15], C. T. C. Wall established that homological invariants are sufficient to determine whether a CW complex has the homotopy type of an n -dimensional CW complex, except possibly when $n = 2$. Recall that a space X has *property $D(n)$* if $H_i(\tilde{X}) = 0$ for $i > n$, where \tilde{X} is the universal cover of X , and if in addition $H^{n+1}(X; \mathcal{M}) = 0$ for all local coefficient systems \mathcal{M} on X . Wall's result is that if $n \neq 2$, then a (finite) CW complex has the homotopy type of a (finite) n -complex if and only if it has property $D(n)$. The assertion that a finite 3-complex has the homotopy type of a finite 2-complex if and only if it has property $D(2)$ has become known as *the $D(2)$ conjecture*. A theorem of M. Dyer (unpublished; a proof can be found in [5]) states that if a group Γ with $H^3(\Gamma; \mathbf{Z}\Gamma) = 0$ has a presentation with

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a relation gap that *realizes the deficiency* of the group (see § 2 for a discussion of deficiency), then the $D(2)$ conjecture is false.

The purpose of this note is to propose a new collection of candidates for presentations with relation gaps. The prime merit of these examples is that one can give a short, transparent and natural proof that the relation modules of the obvious presentations can be generated by one fewer element than one would expect. (Using similar ideas one can create many other examples, but we have resisted the temptation to present these because we do not want to obscure the main idea: the straightforwardness of our examples is what we find most attractive about them.)

Let Q_m be the HNN extension of $\mathbf{Z}/m \times \mathbf{Z}/m$ where the stable letter conjugates the first factor to the second. The obvious presentation of $Q_m * Q_n$ has four generators and four relations, but we shall see that if m and n satisfy a coprime-ness condition then the relation module associated to this presentation requires only three generators. The argument extends easily to arbitrarily many free factors $Q_{m_1} * \cdots * Q_{m_r}$, yielding presentations where the expected relation gap is $r - 1$. Using a result of J. Howie [9] on one-relator products of locally indicable groups, we obtain restrictions on putative 3-relator presentations of $Q_m * Q_n$ (Proposition 3.6), but as yet we have been unable to prove that such presentations do not exist. The difficulty of doing so is discussed in § 5.

In § 4 we pursue the line of attack on the $D(2)$ conjecture via relation gaps discussed above. We follow Harlander's construction to give an explicit description of a 3-complex with only 16 cells that looks homologically like a 2-complex, in the sense that it possesses Wall's property $D(2)$, but that does not have the homotopy type of a finite 2-complex if $Q_m * Q_n$ has a relation gap.

Groups similar to $Q_m * Q_n$ have been studied previously for their interesting presentation theory. For example, the groups used in [8] to show that deficiency is not additive under the operation of free product were of the form $(\mathbf{Z}/m \times \mathbf{Z}/m') * (\mathbf{Z}/n \times \mathbf{Z}/n')$. The groups $G_{m,n} = (\mathbf{Z}/m \times \mathbf{Z}) * (\mathbf{Z}/n \times \mathbf{Z})$ were studied in connection with efficiency by D. Epstein [3], and have come to the fore again recently in the work of K. Gruenberg and P. Linnell [4]. This last work, which is more sophisticated than ours, focuses on the presentation theory of free products of finite groups, but also contains a proof that the relation module associated to the obvious 4-generator, 4-relation presentation of $G_{m,n}$ requires only three generators.

We should mention that the obvious extension of the relation gap problem to finitely generated, rather than just finitely presented, groups is also an interesting problem, and has been resolved by M. Bestvina and N. Brady: in [1], they construct finitely generated groups that are not finitely presented but have finitely generated relation modules. These groups therefore have 'infinite relation gaps'. Naturally, one thinks of trying to build on these examples to find finitely presented groups with relation gaps: we take up this approach in [2] (see also [10]).

2. DEFICIENCY AND ABELIANIZED DEFICIENCY

In this section, we assemble some basic definitions and well-known results for later reference. We write $d(\Gamma)$ for the minimum number of elements needed to generate a group Γ . If Q is a group acting on Γ then we write $d_Q(\Gamma)$ for the minimum number of Q -orbits needed to generate Γ .

Let Γ be a finitely presented group. The *deficiency* of a finite presentation F/R of Γ is $d_F(R) - d(F)$, where F operates on its normal subgroup R by conjugation. (Some authors' definition of deficiency differs from ours by a sign.)

The action of F on R induces by passage to the quotient an action of Γ on the abelianization R_{ab} of R , which makes R_{ab} into a $\mathbf{Z}\Gamma$ -module, called the *relation module* of the presentation. The *abelianized deficiency* of the presentation is $d_\Gamma(R_{\text{ab}}) - d(F)$; this invariant was first studied by K. Gruenberg, under the name of *abelianized defect*.

Lemma 2.1. *If F is free of rank d , then there is an exact sequence of $\mathbf{Z}\Gamma$ -modules*

$$0 \rightarrow R_{\text{ab}} \rightarrow (\mathbf{Z}\Gamma)^d \rightarrow \mathbf{Z}\Gamma \rightarrow \mathbf{Z} \rightarrow 0.$$

Proof. We identify F with the fundamental group of a graph X that has one vertex and d edges. The regular covering \hat{X} of X corresponding to the subgroup $R \subset F$ is the Cayley graph of Γ . The exact sequence in the statement of the lemma is obtained from the cellular chain complex of this covering by inserting the first homology group $H_1(\hat{X}) = R_{\text{ab}}$ on the left as the kernel of the first boundary map. \square

Lemma 2.2. *The deficiency of any finite presentation of Γ is bounded below by the abelianized deficiency, and this in turn is bounded below by $d(H_2(\Gamma)) - \text{rk}(H_1(\Gamma))$, where rk is torsion-free rank.*

Proof. The first part is clear. For the second, let C_* be the cellular complex of the universal cover of a presentation 2-complex of Γ : this is a partial resolution of \mathbf{Z} by free $\mathbf{Z}\Gamma$ -modules. Apply the functor $- \otimes_{\mathbf{Z}\Gamma} \mathbf{Z}$ to C_* . If C_i has rank r_i , then the resulting complex has a free abelian group of the same rank r_i in degree i ; moreover, the homology groups of the new complex are $H_i(\Gamma)$ in degrees $i = 0, 1$, and $\ker \partial_2$ in degree 2. Since $H_2(\Gamma)$ is a quotient of this kernel, one has $d(H_2(\Gamma)) \leq \text{rk}(\ker \partial_2) \leq r_2$, and the result follows. \square

Therefore we can define:

Definition 2.3. The *deficiency* $\text{def}(\Gamma)$ (resp. *abelianized deficiency* $\text{adef}(\Gamma)$) of Γ is the infimum of the deficiencies (resp. abelianized deficiencies) of the finite presentations of Γ .

Remark 2.4. Obviously, if Γ has a presentation of deficiency $d(H_2(\Gamma)) - \text{rk}(H_1(\Gamma))$, then by Lemma 2.2 this presentation realizes the deficiency of the group. In this case, Γ is said to be *efficient*. One knows that inefficient groups exist: R. Swan constructed finite examples in [14], and much later M. Lustig [13] produced the first torsion-free examples. Further examples are given by superperfect groups that are not fundamental groups of homology 4-spheres (see [6] and [7]).

3. THE EXAMPLES

Given letters x and t , let $\rho_n = \rho_n(x, t)$ be the word

$$\rho_n(x, t) = (txt^{-1})x(txt^{-1})^{-1}x^{-n-1}.$$

We will consider the groups $Q_n = \langle x, t \mid \rho_n, x^n \rangle$.

Lemma 3.1. Q_n is isomorphic to the HNN extension

$$(\mathbf{Z}/n \times \mathbf{Z}/n) *_{\phi},$$

where ϕ maps the first factor isomorphically to the second.

Proof. We apply Tietze moves to the given presentation of Q_n , first adding a superfluous generator, and then adding a redundant relator:

$$\begin{aligned} Q_n &= \langle x, t \mid [(txt^{-1}), x], x^n \rangle \\ &= \langle x, b, t \mid b = txt^{-1}, [b, x], x^n \rangle \\ &= \langle x, b, t \mid b = txt^{-1}, [b, x], x^n, b^n \rangle. \end{aligned}$$

□

3.1. Generators for the relation module. Let $q_n = (n+1)^n - 1$ and $c_n = n q_n$.

Lemma 3.2. In $\langle x, t \mid \rho_n \rangle$, one has the equality $[(txt^{-1})^n, x^n] = x^{c_n}$.

Proof. Conjugating x by txt^{-1} in our group n times, we have

$$(txt^{-1})^n x (txt^{-1})^{-n} = x^{q_n+1}.$$

Raising this to the power n gives

$$(txt^{-1})^n x^n (txt^{-1})^{-n} = x^{n(q_n+1)},$$

i.e.

$$[(txt^{-1})^n, x^n] = x^{c_n}.$$

□

Proposition 3.3. Suppose that $(q_m, q_n) = 1$, and let

$$\begin{aligned} \Gamma_{m,n} &= Q_m * Q_n \\ &= \langle x_m, t_m, x_n, t_n \mid \rho_m(x_m, t_m), \rho_n(x_n, t_n), x_m^m, x_n^n \rangle. \end{aligned}$$

Let R_{ab} be the relation module of this presentation. Then R_{ab} is generated as a $\mathbf{Z}\Gamma_{m,n}$ -module by the images of ρ_m , ρ_n and $x_m^m x_n^n$.

Proof. Let S denote the quotient of R_{ab} by the $\mathbf{Z}\Gamma$ -submodule generated by ρ_m and ρ_n , and let $\pi : R \rightarrow S$ be the obvious surjection. It is clear that S is generated as a $\mathbf{Z}\Gamma$ -module by the images of x_m^m and x_n^n , it will be sufficient for us to show that in fact S is a cyclic $\mathbf{Z}\Gamma$ -module, generated by the image of $x_m^m x_n^n$.

Since $x_m^m = (t_m x_m t_m^{-1})^m = 1$ in Q_m , both x_m^m and $(t_m x_m t_m^{-1})^m$ lie in R , and hence their commutator lies in $[R, R]$. On the other hand, by the previous lemma $x_m^{c_m}$ is equal to this commutator modulo ρ_m , so $\pi(x_m^{c_m}) = 1$. Thus the order of $\pi(x_m^m)$ in S divides $q_m = c_m/m$. Similarly, the order of $\pi(x_n^n)$ in S divides q_n . Since q_m and q_n are coprime, it follows that S , which is generated by $\pi(x_m^m)$ and $\pi(x_n^n)$, is actually generated by $\pi(x_m^m)\pi(x_n^n)$ alone, as claimed. □

An entirely similar argument yields:

Proposition 3.4. If $(q_{m_i}, q_{m_j}) = 1$ for $1 \leq i < j \leq r$, then the relation module of

$$\begin{aligned} \Gamma &= Q_{m_1} * \cdots * Q_{m_r} \\ &= \langle x_{m_1}, t_{m_1}, \dots, x_{m_r}, t_{m_r} \mid \rho_{m_1}(x_{m_1}, t_{m_1}), x_{m_1}^{m_1}, \dots, \rho_{m_r}(x_{m_r}, t_{m_r}), x_{m_r}^{m_r} \rangle \end{aligned}$$

is generated as a $\mathbf{Z}\Gamma$ -module by the images of $\rho_{m_1}, \dots, \rho_{m_r}$ and $x_{m_1}^{m_1} \cdots x_{m_r}^{m_r}$.

3.2. An observation on putative 3-relator presentations. Although it certainly does not approach a proof that our groups have relation gaps, the result of this subsection restricts the nature of possible presentations.

Recall that a group is *locally indicable* if each of its non-trivial finitely generated subgroups has the infinite cyclic group as a homomorphic image. We will apply a result about these groups due to J. Howie.

Theorem 3.5 ([9, Theorem 4.2]). *Let A and B be locally indicable groups, and let G be the quotient of $A * B$ by the normal closure of a single element r , not conjugate to an element of A or B . The following are equivalent:*

- a) G is locally indicable;
- b) G is torsion-free;
- c) r is not a proper power in $A * B$.

We call such a group G a *one-relator product* of A and B .

Proposition 3.6. *In the notation of Proposition 3.3, the group $\Gamma_{m,n}$ does not admit a presentation of the form*

$$\langle x_m, t_m, x_n, t_n \mid \rho_m, \rho_n, r \rangle,$$

for any word $r \in \{x_m, t_m, x_n, t_n\}^*$.

Proof. Consider $L = A * B$, where $A = \langle x_m, t_m \mid \rho_m \rangle$ and $B = \langle x_n, t_n \mid \rho_n \rangle$. This is a free product of torsion-free one-relator groups (in particular, of locally indicable groups). Suppose for a contradiction that $\Gamma_{m,n}$ is a one-relator quotient of L by r . Since $\Gamma_{m,n}$ has torsion, Theorem 3.5 implies that either r is conjugate in L to an element of A or B , or else $r = r_0^p$ for some $r_0 \in L$, not itself a proper power. In the second case, the image of r_0 in $\Gamma_{m,n}$ (which we also denote by r_0) is a torsion element, and so is conjugate in $\Gamma_{m,n}$ to an element of $H_m = \langle x_m, t_m x_m t_m^{-1} \rangle \cong \mathbf{Z}/m \times \mathbf{Z}/m$ or $H_n = \langle x_n, t_n x_n t_n^{-1} \rangle \cong \mathbf{Z}/n \times \mathbf{Z}/n$: for the sake of argument, let us say of H_m . Consider M , the one-relator quotient of L by r_0 . The quotient map $L \rightarrow M$ factors through $\Gamma_{m,n}$, so M is isomorphic to $\Gamma_{m,n}$ quotiented by the single additional relator r_0 . Since r_0 is conjugate in $\Gamma_{m,n}$ to an element r'_0 of H_m , it follows that M is also isomorphic to $\Gamma_{m,n}$ quotiented by r'_0 . But it is easy to see that in this last group, the image of x_n is still a torsion element. Thus condition c) of Theorem 3.5 holds for M , but condition b) fails, so we conclude that the hypothesis of the theorem cannot be satisfied: in other words, r_0 must be conjugate in L to an element of A or B .

In either case, after conjugating r if necessary, we can assume that r is contained in one of the factors, A or B : let's say in A . But this means that $\Gamma_{m,n}$ splits as a free product

$$\Gamma_{m,n} = \langle x_m, t_m \mid \rho_m, r \rangle * \langle x_n, t_n \mid \rho_n \rangle,$$

which is absurd since x_n has order n in $\Gamma_{m,n}$. \square

4. A PROPOSED COUNTEREXAMPLE TO THE $D(2)$ CONJECTURE

In this section we associate to each of our groups $\Gamma_{m,n}$ a finite 3-complex with only 16 cells. If $\Gamma_{m,n}$ has a relation gap then the corresponding 3-complex will be a counterexample to the $D(2)$ conjecture.

Theorem 4.1 (Dyer). *Let Γ be a group with $H^3(\Gamma, \mathbf{Z}\Gamma) = 0$. If there is a presentation of Γ that realizes the deficiency of the group and that has a relation gap, then the $D(2)$ conjecture is false.*

A proof of this theorem, including a construction of a counterexample to the $D(2)$ conjecture under the given hypotheses, has been given by Jens Harlander in his fine survey article [5]. In this section we follow Harlander's construction for our groups $\Gamma_{m,n}$.

Let K be the presentation 2-complex associated to our original presentation of $\Gamma = \Gamma_{m,n}$,

$$\Gamma = \langle x_m, t_m, x_n, t_n \mid \rho_m(x_m, t_m), \rho_n(x_n, t_n), x_m^m, x_n^n \rangle.$$

By Proposition 3.3, the relation module R_{ab} associated to this presentation can be generated by three elements, and so there is an exact sequence of $\mathbf{Z}\Gamma$ -modules

$$(*) \quad 0 \rightarrow N \rightarrow (\mathbf{Z}\Gamma)^3 \rightarrow R_{\text{ab}} \rightarrow 0.$$

On the other hand, the cellular chain complex of the universal cover of K is a complex of $\mathbf{Z}\Gamma$ -modules $0 \rightarrow (\mathbf{Z}\Gamma)^4 \xrightarrow{\partial_2} (\mathbf{Z}\Gamma)^4 \xrightarrow{\partial_1} \mathbf{Z}\Gamma \rightarrow \mathbf{Z}$ with zero first homology, so that $\text{im } \partial_2 = \ker \partial_1$, which is exactly the relation module R_{ab} (cf. Lemma 2.1); moreover, $\ker \partial_2$ is isomorphic to $\pi_2(K)$. Therefore we have an exact sequence

$$(\dagger) \quad 0 \rightarrow \pi_2(K) \rightarrow (\mathbf{Z}\Gamma)^4 \xrightarrow{\partial_2} R_{\text{ab}} \rightarrow 0.$$

Applying Schanuel's lemma to the sequences $(*)$ and (\dagger) , we deduce that

$$\pi_2(K) \oplus (\mathbf{Z}\Gamma)^3 \cong N \oplus (\mathbf{Z}\Gamma)^4.$$

Now let L be the 2-complex obtained by taking a 1-point union of K with 3 copies of S^2 . Evidently $\pi_1(L)$ is again isomorphic to Γ , and $\pi_2(L) = \pi_2(K) \oplus (\mathbf{Z}\Gamma)^3$, which we have seen is isomorphic to $N \oplus (\mathbf{Z}\Gamma)^4$. Attach four 3-cells to L to fill the four $(\Gamma$ -orbits of) 2-spheres on the right-hand side of this direct sum, and call the resulting 3-complex M . (The attaching maps of these 3-cells can be described explicitly by tracing through the above algebra.)

Lemma 4.2. *M enjoys property $D(2)$.*

Proof. Since Γ is virtually free, $H^3(\Gamma, \mathbf{Z}\Gamma) = 0$, and it follows from the proof of [5, Theorem 3.5] that M satisfies $D(2)$. \square

Proposition 4.3. *Suppose that $\Gamma_{m,n}$ cannot be presented with fewer than 4 relations on the given generators (i.e. that $\Gamma_{m,n}$ has a relation gap). Then M is a counterexample to the $D(2)$ conjecture.*

Proof. By hypothesis, $\text{def}(\Gamma) = 0$ and $\Gamma = \pi_1(M)$. But

$$\begin{aligned} \chi(M) - 1 &= -4 + (4 + 3) - 4 \\ &= -1 \\ &< 0 \\ &= \text{def}(\Gamma), \end{aligned}$$

and so M cannot have the homotopy type of a finite 2-complex. On the other hand, M is a 3-complex with the $D(2)$ property by the previous lemma. \square

For alternative approaches to the $D(2)$ conjecture, see [11].

5. CONCERNING LOWER BOUNDS ON DEFICIENCY

The nub of both the relation gap problem's difficulty and its attraction is that at present we seem to have no computable invariants that give lower bounds on $d_F(R)$ without giving an identical bound on the number of generators that the relation module requires. There is one sometimes-computable invariant in the literature that at first sight inspires hope in regard to the relation gap problem, namely the deficiency test developed by Martin Lustig in his work on higher Fox ideals [12]. However, in this section we explain why this method cannot help one to establish relation gaps. We are grateful to Ian Leary for helpful comments about this.

Let Γ be a group of type FP_2 . Given a free resolution (C_*, ∂) of \mathbf{Z} over $\mathbf{Z}\Gamma$, finitely generated in degrees ≤ 2 , one defines the *second directed Euler characteristic* of C_* to be $\vec{\chi}_2(C_*) = \text{rk}(C_0) - \text{rk}(C_1) + \text{rk}(C_2)$, and one defines $\vec{\chi}_2(\Gamma)$ by taking the infimum over all such resolutions.

Lustig's deficiency test concerns representations of $\mathbf{Z}\Gamma$ into non-zero unital rings Λ in which left and right inverses coincide.

Let K be the standard 2-complex of a finite presentation \mathcal{P} for Γ , let M be a $\mathbf{Z}\Gamma$ -module with $\pi_2(K) \subset M \subset C_2(\tilde{K})$, and identify $C_2(\tilde{K})$ with the free $\mathbf{Z}\Gamma$ -module on the relators r_j of \mathcal{P} (which index the $\mathbf{Z}\Gamma$ -orbits of 2-cells in \tilde{K}). Fix a finite generating set $\{a_i\}$ for M and express each generator in terms of the basis of $C_2(\tilde{K})$, say $a_i = \sum_j a_{ij}r_j$. Lustig proves that if there is a unital homomorphism $\rho : \mathbf{Z}\Gamma \rightarrow \Lambda$ such that $\rho(a_{ij}) = 0$ for all i and j , then \mathcal{P} realizes the deficiency of Γ .

Proposition 5.1. *If there exists a representation ρ satisfying Lustig's criterion, then the presentation \mathcal{P} does not have a relation gap.*

Proof. What Lustig actually proves is that $\vec{\chi}_2(\Gamma) = \text{def}(\mathcal{P})$. By definition, $\vec{\chi}_2(\Gamma)$ is no greater than the directed Euler characteristic of the resolution in Lemma 2.1, which is the abelianized deficiency of \mathcal{P} . But Lemma 2.2 states that $\text{adef}(\mathcal{P}) \leq \text{def}(\mathcal{P})$. Therefore $\text{adef}(\mathcal{P}) = \text{def}(\mathcal{P})$. \square

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